The Inverse Scattering Transform for a Model of Colomb's plasma with the negative temperature

E. Sh. Gutshabash

Institute Research for Physics, Sankt-Petersburg State University, Russia
e-mail: gutshab@EG2097.spb.edu

Abstract

The boundary problem for a two-dimensional elliptical equation -sinh-Gordon has been investigated. The exact solutions have been found and identities of traces have been proposed. The application of the problem to the model of the Coulomb's plasma with the negative temperature has been considered.

1. Introduction

This paper is devoted to research and the construction of the explicit solutions of the nonlinear elliptical equation -sinh-Gordon:

$$\triangle u = -4 \sinh u, \tag{1.1}$$

where $\triangle$ is two-dimensional Laplas's operator.

On the one hand, it is of a great mathematical interest due to is describes the immersion of a negative curvature surface in the three-dimensional Euclidean space [1], and on the other hand it has a number of physical applications. In particular, it arises as the model of two-dimensional Colomb's system at the negative temperatures [2,3].

L.Onzager appears to have been first who introduced a conception of negative temperatures in the context of a problem of a vortical line description. Further the conception turned out to be connected with so-called anomalous systems consisting of weakly interacting particles, spins and so on (see, for example [4]) where an energetic spectrum is bounded above and some maximal value of energy is available. In this case a statistical sum of the system is $\sum_i g(E_i)\exp(-E_i/T)$ under assumption that $g(E_i) \sim \exp(-aE_i)$, $E_i \to \infty$, $a > 0$, where $g(E_i)$ is a density of a statements number , and E_i is an energetic level that can be finite both at $T > 0$ and $T < 0$ (T is absolute temperature). Assuming that $a = \infty$ and hence $g(E_i) = 0$ at $E_i > E_{\max}$, it is not difficult to see that the statistical sum changes into a finite series and has a finite value at $T \in (-\infty, \infty)$. Then it is obvious that the density of the energetic levels that is proportional to Boltsmann's factor $\exp(-E_i/T)$, at $T > 0$ drops as the energy increases.

In particular, at $T \to \infty$ $\exp(-E_i/T) \to 1$, i.e. all levels proved to be distributed uniformly. Another picture is at $T \in (-\infty, 0)$, in this case Boltsmann's factor is an increasing function of E_i and there exists an phenomenon of an inversion of a levels distribution with the density at the up levels becoming more than at down ones. It is clear, that the equation (1.1) involving the difference of Boltsmann's factors for the one-charged particles in its right side exactly corresponds to the situation of a negative temperatures that was described above.

We now turn to a statement the boundary problem. Let us suppose that the function

$u(x,y)$ determined on the half-plane $\mathbb{R}^2_+ = \{(x,y) : x \in (-\infty,\infty),\ y \geq 0\}$, is the real-valued and enough smooth, $u(x,0)$, $u_y(x,0)$ are boundary values of the desired initial function and its normal derivative correspondently. Also we assume that one of these condition is known, but there is some relation of the condition to one another that will by some given below (a nonlinear analog of the condition of the third kind).

Moreover, we will assume, that

$$u(x,0) \to 0 \quad at \ |x| \to \infty. \tag{1.2}$$

Equation (1.1) is integrable and can be represented in the form of the compatibility condition of the linear matrix system:

$$\Psi_x(x,y,\lambda) = U(x,y,\lambda)\Psi(x,y,\lambda), \tag{1.3a}$$

$$\Psi_y(x,y,\lambda) = V(x,y,\lambda)\Psi(x,y,\lambda). \tag{1.3b}$$

Here $\Psi(x,y,\lambda)$ is a matrix-valued (2×2) function, $\lambda \in \mathbb{C}$ is a spectral parameter, U and V will look like:

$$U(\lambda) \equiv U(x,y,\lambda) = i[\frac{2}{\lambda^2-1} + \frac{\lambda+1}{2(\lambda-1)}(\cosh u - 1)]\sigma_3 + \frac{\lambda+1}{2(\lambda-1)}\sinh u e^{-2ix\sigma_3}\sigma_2 +$$

$$+\frac{u_z}{2}e^{-2ix\sigma_3}\sigma_1,$$

$$V(\lambda) \equiv V(x,y,\lambda) = [\frac{2\lambda}{\lambda^2-1} + i\frac{\lambda+1}{2(\lambda-1)}(\cosh u - 1)]\sigma_3 + \frac{i(\lambda+1)}{2(\lambda-1)}\sinh u e^{-2ix\sigma_3}\sigma_2 +$$

$$+\frac{iu_z}{2}e^{-2ix\sigma_3}\sigma_1.$$

It should be notice that a rather nontrivial and complex form of the gauge of the operators U and V is connected with the gauge equivalence of the equation (1.1) and two-dimensional isotropic Heisenberg's ferromagnet, which was proved earlier [5,6]. The choice ± 1 as the poles is necessary to obtain a convenient system of contours in solving Riemann's problem (see below).

2. Associated linear problem

Let us $\Psi^{\pm}(x,\lambda)$ are the solutions of the (1.3) with the conditions ($k(\lambda) = 2/(\lambda^2-1)$):

$$\Psi^{\pm}(x,\lambda) \to \exp(ik(\lambda)x\sigma_3)(1+o(1)), \quad x \to \pm\infty. \tag{2.1}$$

Let us determine a transition matrix $T(\lambda) \equiv T(\lambda, y)$ ($\lambda^2 = \bar{\lambda}^2$):

$$\Psi^-(x,\lambda) = \Psi^+(x,\lambda)T(\lambda), \ T(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}. \tag{2.2}$$

The matrix $T(\lambda)$ is unimodular: $\det T(\lambda) = 1$.

Assuming $\phi^{\pm}(x,\lambda) = \Psi^{\pm}(x,\lambda)\exp(-ik(\lambda)x\sigma_3)$, from (1.3a) we obtain:

$$\phi^{\pm}(\lambda) = e^{-ix\sigma_3}e^{(u/2)\sigma_1}\sigma_2 e^{-ix\sigma_3}\bar{\phi}^{\pm}(-\bar{\lambda})\sigma_2, \;\; \lambda^2 = \bar{\lambda}^2, \tag{2.3}$$

$$\phi^{\pm}(\lambda) = e^{-2ix\sigma_3}\sigma_1\phi^{\pm}(\frac{1}{\lambda})\sigma_1, \;\; \lambda^2 = \bar{\lambda}^2. \tag{2.4}$$

From these relations it follows the relations of symmetry

$$T(\lambda) = \sigma_2\bar{T}(-\bar{\lambda})\sigma_2, \;\; \lambda^2 = \bar{\lambda}^2, \tag{2.5}$$

$$T(\lambda) = \sigma_1\bar{T}(\frac{1}{\lambda})\sigma_1. \tag{2.6}$$

Then we have

$$T(\lambda) = \sigma_3\bar{T}(-\frac{1}{\bar{\lambda}})\sigma_3, \;\; \lambda^2 = \bar{\lambda}^2. \tag{2.7}$$

We find a dependence of scattering data from the variable y. For this purpose let us turn to the system (1.3b). On fulfilling standard operations and taking $l(\lambda) = 2\lambda/(\lambda^2 - 1) = \lambda k(\lambda)$, we obtain:

$$a(\lambda,0) = a(\lambda,y), \quad b(\lambda,y) = b(\lambda,0)e^{2l(\lambda)y}, \tag{2.8}$$

$$c(\lambda,y) = c(\lambda,0)e^{-2l(\lambda)y}, \quad d(\lambda,0) = d(\lambda,y). \tag{2.9}$$

It follows from these relations that the coefficients $a(\lambda)$ and $d(\lambda)$ act as the produced functional of "the integrals of movement". Alternatively, from the requirement of finiteness of the coefficients $b(\lambda)$ and $c(\lambda)$ it follows, that it is necessary to put $b(\lambda,0)$ equal to zero at

$$\lambda > 1, \quad -1 < \lambda < 0, \tag{2.10}$$

and put $c(\lambda,0)$ equal to zero at

$$0 < \lambda < 1, \quad \lambda < -1. \tag{2.11}$$

These inequalities determine boundaries of the zones. Domains in the real axis that are additional to these zones are continuous spectrum. There are no such restrictions at $\lambda = -\bar{\lambda}$, so the continuous spectrum can be find in the imaginary axis.

Taking $U(x,y,\lambda) = ik(\lambda)\sigma_3 + Q(x,y,\lambda)$, where

$$Q(x,y,\lambda) = \begin{pmatrix} \frac{i}{2}\frac{\lambda+1}{\lambda-1}(\cosh u - 1) & (-\frac{i}{2}\frac{\lambda+1}{\lambda-1}\sinh u + \frac{u_z}{2})e^{-2ix} \\ (\frac{i}{2}\frac{\lambda+1}{\lambda-1}\sinh u + \frac{u_z}{2})e^{2ix} & -\frac{i}{2}\frac{\lambda+1}{\lambda-1}(\cosh u - 1) \end{pmatrix},$$

and introducing the columns $\phi_1^{\pm}$, $\phi_2^{\pm}$ of the matrix $\phi^{\pm}$, from (1.3a) we will obtain Volterra's integrable equations of the direct problem:

$$\phi_1^{-}(x,\lambda) = e_1 + \int d\xi g_1^{-}(x-\xi,\lambda)Q(\xi,\lambda)\phi_1^{-}(\xi,\lambda), \tag{2.12}$$

$$\phi_2^{-}(x,\lambda) = e_2 + \int d\xi g_2^{-}(x-\xi,\lambda)Q(\xi,\lambda)\phi_2^{-}(\xi,\lambda). \tag{2.13}$$

In these equations $e_1 = (1,0)^T$, $e_2 = (0,1)^T$ and $g^-_{1,2}(x,\lambda)$ are Green's "initial" functions:

$$g_1^-(x,\lambda) = \theta(x) diag(1,\ e^{-2ik(\lambda)x}), \quad g_2^-(x,\lambda) = \theta(x) diag(e^{2ik(\lambda)x},\ 1). \tag{2.14}$$

From (2.12)-(2.13) it follows that

$$\phi_1^-(x,\lambda) \in H(\Omega_1 \cup \Omega_3), \quad \phi_2^-(x,\lambda) \in H(\Omega_2 \cup \Omega_4), \tag{2.15}$$

where $H(\Omega_i)$ is the class of functions ($\Omega_i = \{\lambda : (i-1)\pi/2 < \arg\lambda < i\pi/2\}, i = 1, ...4$), that are analytical in the domain Ω_i.

Completely analogously we can show that

$$\phi_1^+(x,\lambda) \in H(\Omega_2 \cup \Omega_4), \quad \phi_2^+(x,\lambda) \in H(\Omega_1 \cup \Omega_3). \tag{2.16}$$

From (2.12)-(2.13) we can easy obtain the integrable representations for the scattering data:

$$a(\lambda) = 1 + \int d\xi [Q_{11}(\xi,\lambda)\phi_{11}^-(\xi,\lambda) + Q_{12}(\xi,\lambda)\phi_{21}^-(\xi,\lambda)], \tag{2.17}$$

$$b(\lambda) = \int d\xi [Q_{11}(\xi,\lambda)\phi_{12}^-(\xi,\lambda) + Q_{12}(\xi,\lambda)\phi_{22}^-(\xi,\lambda)] e^{-2ik(\lambda)\xi}, \tag{2.18}$$

and hence

$$a(\lambda) \in H(\Omega_1 \cup \Omega_3). \tag{2.19}$$

From (2.18) and the similar expression for $c(\lambda)$ it follows that generally speaking, $b(\lambda)$ and $c(\lambda)$ don't admit an analytical extension from both the real axis and imaginary one. On assuming $y = 0$, in (2.18), accordingly to (2.10), we obtain the equality:

$$\int d\xi [Q_{11}(\xi,0,\lambda)\phi_{12}^-(\xi,0,\lambda) + Q_{12}(\xi,0,\lambda)\phi_{22}^-(\xi,0,\lambda)] e^{-2ik(\lambda)} = 0, \tag{2.20}$$

that connects the boundary values of the desired function and its normal derivative and acts as the boundary condition (in term of the spectral representation).

Let us introduce a matrix complex-valued function $\Omega = \Omega(x,y)$, that can be defined as

$$\Omega(x,y) = \lim_{|\lambda|\to\infty} \Psi^-(x,y,\lambda). \tag{2.21}$$

From (2.21), (2.3) it follows, that the expression

$$\Omega = e^{-ix\sigma_3} e^{\frac{u}{2}\sigma_1} \sigma_2 e^{-ix\sigma_3} \bar{\Omega} \sigma_2 \tag{2.22}$$

is fulfilled. This relation will be used later.

Matrix Ω obeys the equation:

$$\Omega_x = U(x,y,\infty)\Omega, \tag{2.23}$$

and, in addition, $\det\Omega(x,y) = 1$. We suppose also that it has asymptotics:

$$\Omega_0 = \lim_{x\to+\infty} \Omega(x) = (-1)^N e^{\frac{i\alpha_0}{2}\sigma_3}, \tag{2.24}$$

$$\lim_{x \to -\infty} \Omega(x) = I, \tag{2.25}$$

where α_0 is some parameter, $\alpha_0 \not\equiv 0$, and the sense of the integer number N will be clear further. The choice of the asymptotics is in agreement with (1.3a), (2.23) and is stimulated by the gauge equivalence of boundary problems.

From (2.2), (2.23), (2.24) we have:

$$T(\infty) = (-1)^N \exp(\frac{i\alpha_0}{2}\sigma_3), \quad T(0) = (-1)^N \exp(-\frac{i\alpha_0}{2}\sigma_3), \tag{2.26}$$

and from (2.10) and the similar expression for $c(\lambda)$ it follows that

$$b(\pm 1 - 0) = c(\pm 1 + 0) = 0. \tag{2.27}$$

3. Dispersion relations and identities of traces.

The results are obtained in the previous sections allows to write down the dispersion relations and to obtain identities of traces as well.

From (2.17) we have:

$$a(\lambda) = (-1)^N e^{\frac{i\alpha_0}{2}}[1 + O(\frac{1}{|\lambda|})], \;\; |\lambda| \to \infty, \;\; \lambda \in \Omega_1 \cup \Omega_3. \tag{3.1}$$

Similarly it can be shown

$$d(\lambda) = (-1)^N e^{-\frac{i\alpha_0}{2}}[1 + O(\frac{1}{|\lambda|})], \;\; |\lambda| \to \infty, \;\; \lambda \in \Omega_2 \cup \Omega_4. \tag{3.2}$$

We assume now that the coefficient $a(\lambda)$ has finite number of simple zeros in domains Ω_1 and Ω_3, with the quantity of the zeros being same and equal to N in both domains. Let us also assume that the zeros don't belong to the continuous spectrum (may be except of the zones accordingly to the formula (2.10)). Moreover, we will consider that $a(\lambda) = a_s(\lambda)a_c(\lambda)$, where $a_s(\lambda)$ is given by the contribution of the discrete part of the spectrum ("solitonic" part) and $a_c(\lambda)$ - to the continuous part spectrum. Taking into account (2.2), (2.10)-(2.11) and equality $|a_s(\lambda)|^2 = 1$ at $\lambda = \bar{\lambda}$, we will have

$$\ln a_c(\lambda) + \ln d_c(\lambda) = 0, \;\; \lambda = \bar{\lambda},$$
$$\ln a_c(\lambda) + \ln d_c(\lambda) = \ln(1 - |b(\lambda)|^2), \;\; \lambda = -\bar{\lambda}.$$

These equalities assign the bound of piecewise analytical functions $\ln a_c(\lambda)$ and $\ln d_c(\lambda)$ in the continuous spectrum. On applying the Cochy's formula to the function $\ln a_c(\lambda) \in H(\Omega_1 \cup \Omega_3)$ and taking into account both (2.26) and two previous relations, we obtain

$$\ln a_c(\lambda) = -\int_{-\infty}^{\infty} \frac{d\mu}{2\pi i} \frac{\ln(1 - |b(i\mu)|^2)}{\mu + i\lambda}\, sign(\mu). \tag{3.3}$$

Now let us consider the "soliton" part of the coefficients $a(\lambda)$: $a_s(\lambda)$. Let $a_s(\lambda_n) = 0$, where $\lambda_n \in \Omega_1$. Then in a view of the involution $\bar{a}(\lambda_n) = a(-1/\bar{\lambda}_n)e^{i\alpha_0}$ we have: $a(-1/\bar{\lambda}) = 0$, $-\bar{\lambda}^{-1} \in \Omega_3$. From this and with consideration for (2.6) we will find the expression for $a_s(\lambda)$:

$$a_s(\lambda) = (-1)^N e^{\frac{i\alpha_0}{2}} \prod_{n=1}^{N} \frac{(\lambda - \lambda_n)(\lambda + \frac{1}{\bar{\lambda}_n})}{\lambda + \bar{\lambda}_n)(\lambda - \frac{1}{\lambda_n})} \,. \tag{3.4}$$

Combining the equation (3.3) and (3.4), we will finally obtain

$$a(\lambda) = (-1)^N e^{\frac{i\alpha_0}{2}} \prod_{n=1}^{N} \frac{(\lambda - \lambda_n)(\lambda + \frac{1}{\bar{\lambda}_n})}{\lambda + \bar{\lambda}_n)(\lambda - \frac{1}{\lambda_n})} \times$$

$$\times \exp\left[-\int_{-\infty}^{\infty} \frac{d\mu}{2\pi i} \frac{\ln(1 - |b(i\mu)|^2)}{\mu + i\lambda} \, sign(\mu)\right]. \tag{3.5}$$

From the latter with consideration for (2.26) it follows:

$$e^{-i\alpha_0} = \prod_{n=1}^{N} \frac{\lambda_n^2}{\bar{\lambda}_n^2} \exp\left[-\int_{-\infty}^{\infty} \frac{d\mu}{2\pi i} \frac{\ln(1 - |b(i\mu)|^2}{\mu}\right]. \tag{3.6}$$

In particular, at $b(\mu) = 0$, i.e. in the "solitonic" sector of the problem we have:

$$e^{-i\alpha_0} = \prod_{n=1}^{N} \frac{\lambda_n^2}{\bar{\lambda}_n^2} \,. \tag{3.7}$$

It follows here from that at $\alpha_0 = 0$ and $N = 1$ either $\lambda_1 = \bar{\lambda}_1$ or $\lambda_1 = -\bar{\lambda}_1$. In the first case the eigenfunction proves to be non-localized and the solution doesn't fit to the asymptotic (2.1). In the latter case ($\lambda_1 = -\bar{\lambda}_1$) $a_s(\lambda_1) = 1$; means that discrete eigenvalue is in the continuous spectrum (the situation that is beyond the consideration). Hence in the reflectionless case at $\alpha_0 = 0$ the minimal value of N is equal two. This explains the cause of introducing the parameter α_0 in (2.24).

Assuming $\lambda_n = \rho_n e^{i\theta_n}$, $\rho_n > 0$, $\theta_n \in (0, \pi/2)$, in (3.7), we will obtain $\sum_{i=1}^{N} \theta_n = -\alpha_0/4 + \pi n/2$, $n = 0, \pm 1, \pm 2, \ldots$; in particular, at $N = 1$: $\theta_1 = -\alpha_0/4 + \pi k/2$, where k is the number to fulfill the inequalities $0 < -\alpha_0/4 + \pi k/2 < \pi/2$.

Let $a_s(\lambda) = 1$, i.e. discrete spectrum in the system is unavailable. In this case from (3.6) we have:

$$\alpha_0 = -\int_{0}^{\infty} \frac{d\mu}{\pi} \frac{\ln(1 - |b(i\mu)|^2)}{\mu} \,. \tag{3.8}$$

From (3.5) and (3.8) it follow, that

$$T(\pm 1) = 1 \,. \tag{3.9}$$

It should be noticed, that the expression of the type of (3.5) holds for the coefficient $d(\lambda)$ also, however we will not discuss it here.

Let us turn to deducing the identities of traces. Do to the matrix Ω (the relation (2.23)) can't be determined explicitly the equalities in the points zero and infinity can't be obtained (because we don't know solutions of the Riccati's equation in this cases). Therefore we will restrict ourselves by the consideration of the relations near points of the poles divisors.

Assuming $\ln a(\lambda) = \sum_{p=0}^{\infty} a_p^{(\pm 1)} (\lambda \pm 1)^p, \ \lambda \to \pm 1$, from (3.5) we have: $a_p^{(1)} = -a_p^{(-1)}$, where p is a even number, and $a_p^{(1)} = \bar{a}_p^{(-1)}$, where p is an odd number. These equalities being the consequence of the spectral structure of the problem permit to control the validity of intermediate calculations and the final result.

From (1.3a) using the standard way of calculations one can obtain Riccati's equation for the function $F(x, \lambda) = \phi_{21}^-(x, \lambda)/\phi_{11}^-(x, \lambda)$:

$$F_x + Q_{12} F^2 = (Q_{22} - Q_{11} - 2ik(\lambda))F + Q_{21}. \tag{3.10}$$

Also from the equation for the function ϕ_1^-, in view of (2.2), it follows that

$$\ln a(\lambda) = \int_{-\infty}^{\infty} dx [Q_{11}(x, \lambda) + Q_{12}(x, \lambda) F(x, \lambda)]. \tag{3.11}$$

Using explicit form of the matrix elements Q_{ij}, at the vicinity of the point $\lambda = -1$ we will obtain $(F(x, \lambda) = \sum_{s=1}^{\infty} F_s^{(-1)} (\lambda + 1)^s))$: $F_1^{(-1)} = (iu_z/2)e^{2ix}$. Then accordingly to (3.11) $\ln a(\lambda) = 0, \ \lambda \to -1$, which is in agreement with (3.9). In the next order ($\sim (\lambda + 1)$), considering (3.10), (3.11) and (3.5), we will have:

$$\begin{aligned} 1 + \frac{\rho_1^2 - 1}{\rho_1^2 + 2\rho_1 \cos\theta_1 + 1} + \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \frac{(\mu^2 - 1)\ln(1 - |b(i\mu)|^2)}{(\mu^2 - 1)^2 + 4\mu^2}\, sign(\mu) = \\ = \frac{1}{16} \int_{-\infty}^{\infty} dx u_x(x, 0) u_y(x, 0), \end{aligned} \tag{3.12}$$

$$\begin{aligned} -\frac{\rho_1 \sin\theta_1}{\rho_1^2 + 2\rho_1 \cos\theta_1 + 1} + \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \frac{\mu \ln(1 - |b(i\mu)|^2)}{(\mu^2 - 1)^2 + 4\mu^2}\, sign(\mu) = \\ = -\frac{1}{4} \int_{-\infty}^{\infty} dx [(\cosh u(x, 0) - 1) - \frac{1}{8}(u_x^2(x, 0) - u_y^2(x, 0))], \end{aligned} \tag{3.13}$$

where $\lambda_1 = \rho_1 e^{i\theta_1}, \ \theta_1 \in (0, \pi/2)$. Thus, the continuous spectrum of the problem contributes the left parts of the identities of traces. It should be noted that the similar character the identities have also in the neighbourhood of the point $\lambda = 1$.

4. Equations of the inverse problem and formulas for reconstruction.

It follows from the analysis of properties of the operator spectrum of an associated linear problem and analytical properties of Jost's solutions performed above that the continuous spectrum of the problem is in "the cross": $\{\lambda : \lambda_R = 0 \cup \lambda_I = 0\}$, whereas matrix columns of these solutions admit the analytical extension to certain domains of the plane of a complex spectral parameter λ. This allows using Riemann's technique problem. However there exists some additional difficulty here: from (2.21) and the determination of the matrix ϕ^- it follows, that $\phi^-(x, \infty) = \Psi(x, \infty) = \Omega(x, y)$, with the matrix $\Omega(x, y)$ being unknown. This leads to Riemann's local problem on an analytical factorization appears to be characterized by noncanonical conditions at the infinity in this case. Also some difficulties arise in searching for formulas of the reconstruction. This section is devoted to these problems.

As the method of solution of the inverse problem we choose the method of Riemann's vector problem. From (2.2) and the connection $\Psi^{\pm}$ and $\phi^{\pm}$ we obtain:

$$\phi^{-}(x,\lambda) = \phi^{+}(x,\lambda)T_1(x,\lambda)\,, \tag{4.1}$$

where

$$T_1(x,\lambda) = \exp(ik(\lambda)x\sigma_3)T(\lambda)\exp(-ik(\lambda)x\sigma_3).$$

We use one of the equalities following from (4.1)

$$\frac{\phi_1^{+}(x,\lambda)}{d(\lambda)} = \phi_1^{-}(x,\lambda) - \phi_2^{-}(x,\lambda)\frac{c(\lambda)}{d(\lambda)}e^{-2ik(\lambda)x}\,. \tag{4.2}$$

Now let us suppose that $\lambda_n \in \Omega_1$ is such that $a(\lambda_n) = 0$ and also $-1/\bar{\lambda}_n \in \Omega_3$ and $a(-1/\bar{\lambda}_n) = 0$. Then the vector-functions $\Psi_1^{-}(x,\lambda)$ and $\Psi_2^{+}(x,\lambda)$ proved to be linear-dependent at these points, from here we have:

$$\phi_1^{-}(x,\lambda_n) = -\frac{1}{b_n}\phi_2^{+}(x,\lambda_n)e^{-2ik(\lambda_n)x}, \tag{4.3}$$

$$\phi_1^{-}(x,-\frac{1}{\bar{\lambda}_n}) = \frac{1}{\bar{b}_n}\phi_2^{+}(x,-\frac{1}{\bar{\lambda}_n})e^{-2ik(-\frac{1}{\bar{\lambda}_n})x}, \tag{4.4}$$

where $\phi_1^{-}(\lambda_n)$, $\phi_2^{+}(\lambda_n)$, $\phi_1^{-}(-\bar{\lambda}_n^{-1})$, $\phi_2^{+}(-\bar{\lambda}_n^{-1})$ are eigenfunctions of the discrete spectrum, b_n are the coefficients of transition.

Taking into account that $\phi_1^{-}(\lambda) \to \Omega_1^1(x,y)$, $|\lambda| \to \infty$, $\lambda \in \Omega_1 \cup \Omega_3$, and $\phi_1^{+}(\lambda)/d(\lambda) \to \Omega_1^1(x,y)$, $\lambda \in \Omega_2 \cup \Omega_4$, where $\Omega_1^1 = \Omega(x,y)e_1$, and applying Cochy's formulas to the vector-column $(\phi_1^{-} - \Omega_1^1)$ that is analytical in Ω_1 and also using (4.2)-(4.4), we obtain the following equality:

$$\begin{aligned}\phi_1^{-}(\lambda) = \Omega_1^1 - i\sum_{n=1}^{N}\frac{\beta_n}{\lambda_n^2(\lambda - \frac{1}{\lambda_n})}e^{-2ik(\frac{1}{\lambda_n})x}\chi(x)\bar{\phi}_1^{-}(-\frac{1}{\bar{\lambda}_n}) +\\ +i\sum_{n=1}^{N}\frac{\bar{\beta}_n}{\lambda+\bar{\lambda}_n}\chi(x)\bar{\phi}_1^{-}(\lambda_n) + \int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\bar{b}(\mu)}{\bar{a}(\mu)}\frac{\chi(x)}{\mu+\lambda}e^{-2ik(\mu)x}\bar{\phi}_1^{-}(\mu)sign(\mu) -\\ -\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\bar{b}(i\mu)}{\bar{a}(i\mu)}\frac{\chi(x)}{\mu+i\lambda}e^{-2ik(i\mu)x}\bar{\phi}_1^{-}(i\mu)sign(\mu),\end{aligned} \tag{4.5}$$

where $\chi(x) = e^{-ix\sigma_3}\sigma_2 e^{-iu/2\sigma_1}e^{-ix\sigma_3}$, $\beta_n(y) = b_n/a'(\lambda_n) = \beta_n(0)e^{2l(\lambda_n)y}$. The equation is not closed equation for the column $\phi_1^{-}(\lambda)$ yet due to it involves an unknown vector-function Ω_1^1. Therefore to obtain the necessary equation we will use (2.22) rewritten in the form:

$$\Omega^{-1}e^{-ix\sigma_3}\sigma_2 e^{-\frac{u}{2}\sigma_1}e^{-x\sigma_3}\bar{\Omega} = \sigma_2\,. \tag{4.6}$$

Substituting (4.6) into (4.5) and putting

$$\phi_1^{-}(\lambda) = \Omega(x,y)\tilde{\phi}_1^{-}(\lambda), \tag{4.7}$$

we obtain the following system of singular integral equations for the function $\tilde{\phi}_1^-(\lambda)$:

$$\tilde{\phi}_1^-(\lambda) = e_1 + i\sum_{n=1}^{N} \frac{\bar{\beta}_n e^{-2ik(\bar{\lambda}_n)x}}{\lambda + \bar{\lambda}_n}\sigma_2\bar{\tilde{\phi}}_1^-(\bar{\lambda}_n)$$

$$-i\sum_{n=1}^{N}\frac{\beta_n e^{-2ik(\frac{1}{\lambda_n})x}}{\lambda_n^2(\lambda - \frac{1}{\lambda_n})}\sigma_2\bar{\tilde{\phi}}_1^-(-\frac{1}{\bar{\lambda}_n}) + \int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\bar{b}(\mu)}{\bar{a}(\mu)}\frac{e^{-2ik(\mu)x}}{\mu+\lambda}\sigma_2\bar{\bar{\phi}}_1^-(\mu)sign(\mu)-$$

$$-\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\bar{b}(i\mu)}{\bar{a}(i\mu)}\frac{e^{-2ik(i\mu)x}}{\mu+i\lambda}\sigma_2\bar{\tilde{\phi}}_1^-(i\mu)sign(\mu), \tag{4.8}$$

$$\tilde{\phi}_1^-(\lambda_p) = e_1 + i\sum_{n=1}^{N} \frac{\bar{\beta}_n e^{-2ik(\bar{\lambda}_n)x}}{\lambda_p + \bar{\lambda}_n}\sigma_2\bar{\tilde{\phi}}_1^-(\bar{\lambda}_n)-$$

$$-i\sum_{n=1}^{N}\frac{\beta_n e^{-2ik(\frac{1}{\lambda_n})x}}{\lambda_n^2(\lambda_p - \frac{1}{\lambda_n})}\sigma_2\bar{\tilde{\phi}}_1^-(-\frac{1}{\bar{\lambda}_n}) + \int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\bar{b}(\mu)}{\bar{a}(\mu)}\frac{e^{-2ik(\mu)x}}{\mu+\lambda_p}\sigma_2\bar{\bar{\phi}}_1^-(\mu)sign(\mu)-$$

$$-\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\bar{b}(i\mu)}{\bar{a}(i\mu)}\frac{e^{-2ik(i\mu)x}}{\mu+i\lambda_p}\sigma_2\bar{\tilde{\phi}}_1^-(i\mu)sign(\mu), \tag{4.9}$$

$$\tilde{\phi}_1^-(-\frac{1}{\bar{\lambda}_p}) = e_1 + i\sum_{n=1}^{N} \frac{\bar{\beta}_n e^{-2ik(\bar{\lambda}_n)x}}{\bar{\lambda}_n - 1/\bar{\lambda}_p}\sigma_2\bar{\tilde{\phi}}_1^-(\bar{\lambda}_n)+$$

$$+i\sum_{n=1}^{N}\frac{\beta_n e^{-2ik(\frac{1}{\lambda_n})x}}{\lambda_n^2(\frac{1}{\bar{\lambda}_p} + \frac{1}{\lambda_n})}\sigma_2\bar{\tilde{\phi}}_1^-(-\frac{1}{\bar{\lambda}_n}) + \int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\bar{b}(\mu)}{\bar{a}(\mu)}\frac{e^{-2ik(\mu)x}}{\mu-\frac{1}{\bar{\lambda}_p}}\sigma_2\bar{\bar{\phi}}_1^-(\mu)sign(\mu)-$$

$$-\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\frac{\bar{b}(i\mu)}{\bar{a}(i\mu)}\frac{e^{-2ik(i\mu)x}}{\mu-i\frac{1}{\bar{\lambda}_p}}\sigma_2\bar{\tilde{\phi}}_1^-(i\mu)sign(\mu). \tag{4.10}$$

The equations (4.9), (4.10) are obtained from (4.8) at $\lambda = \lambda_p$ and $\lambda = -1/\bar{\lambda}_p$ respectively. On the right-hand sides of these equations it is necessary to take into account the availability of the zones (2.10).

The system (4.8)-(4.10) is the system of the equations for the eigenfunctions of the continuous and discrete spectrum. Considering the system is written for the "gauge" vector-functions, the necessary formulas of the reconstruction should be written in terms of the same function.

Using (1.3a), (4.7), we obtain the differential equation for the matrix $\tilde{\phi}^- \equiv \tilde{\phi}^-(x,\lambda)$:

$$\tilde{\phi}_x^- = -ik(\lambda)\tilde{\phi}^-\sigma_3 + \frac{1}{\lambda-1}\Omega^{-1}[i(\cosh u - 1)\sigma_3 + \sinh u e^{-2ix\sigma_3}\sigma_2]\Omega\tilde{\phi}^-+$$

$$+ik(\lambda)\Omega^{-1}\sigma_3\Omega\tilde{\phi}^-. \tag{4.11}$$

In the neighborhoods of the point $\lambda = 1$, setting $\tilde{\phi}^- = \sum_{k=0}^{k=\infty}\tilde{\phi}_k(\lambda-1)^k$ ($\tilde{\phi}_0^- \to I,\ x \to -\infty$), we find:

$$\tilde{\phi}_0^- = \Omega^{-1}\begin{pmatrix} 1 & \tanh\frac{u}{2}e^{-2ix} \\ \tanh\frac{u}{2}e^{2ix} & 1 \end{pmatrix}. \tag{4.12}$$

Now we parameterize the unknown matrix Ω by four complex functions of variables $x,\ y$:

$$\Omega(x,y) = \left(\begin{array}{cc} \alpha(x,y) & \beta(x,y) \\ \gamma(x,y) & \delta(x,y) \end{array} \right). \tag{4.13}$$

Then the relation (4.6) gives 4 algebraic connections:

$$\alpha = \bar{\delta}\cosh\frac{u}{2} - \bar{\beta}\sinh\frac{u}{2}e^{-2ix}, \quad \beta = \bar{\alpha}\sinh\frac{u}{2}e^{-2ix} - \bar{\gamma}\cosh\frac{u}{2},$$
$$\gamma = -\bar{\beta}\sinh\frac{u}{2} + \bar{\delta}\cosh\frac{u}{2}e^{2ix}, \quad \delta = \bar{\alpha}\cosh\frac{u}{2} - \bar{\gamma}\sinh\frac{u}{2}e^{2ix}.$$

In a view of the relation and from (4.12) we obtain ($\tilde{\phi}^0 = \tilde{\phi}_0$):

$$\tilde{\phi}^0 = \frac{1}{\cosh\frac{u}{2}}\bar{\Omega}^T, \tag{4.14}$$

where $\det\tilde{\phi}^0 = (\cosh\frac{u}{2})^{-2}$. Combining (4.12), (4.14) and (4.6), we will have:

$$\cosh\frac{u(x,y)}{2} = [|\tilde{\phi}^0_{11}|^2 + |\tilde{\phi}^0_{21}|^2]^{-1} = [|\tilde{\phi}^0_{12}|^2 + |\tilde{\phi}^0_{22}|^2]^{-1}, \tag{4.15}$$

or

$$\cosh\frac{u(x,y)}{2} = [< {\tilde{\phi}^0_1}^T, \tilde{\phi}^0_1 >_2]^{-1} = [< {\tilde{\phi}^0_2}^T, \tilde{\phi}^0_2 >_2]^{-1}, \tag{4.16}$$

where the symbol $<,>_2$ denotes the scalar product of the vectors in $\mathbb{C}^2$. Another equivalent representation of (4.16) is

$$\cosh\frac{u(x,y)}{2} = \{\frac{1}{2}Tr({\tilde{\phi}^0}^T\bar{\tilde{\phi}}^0)\}^{-1}. \tag{4.17}$$

Analogously we can obtain one more formulas of the reconstruction:

$$\sinh\frac{u(x,y)}{2} = \frac{2\tilde{A}e^{-2ix}}{1+\sqrt{1-4\tilde{A}^2e^{-4ix}}}, \tag{4.18}$$

where $\tilde{A} =< (((\phi^-)^0)_1)^T, (((\phi^-))^0)_2 >_2$, $\tilde{A} \to 0$ at $|x| \to \infty$, and the requirement of reality of the potential leads to the value $\tilde{A}e^{-2ix}$ is to be real. It is easy to prove the latter using (4.14) and the definition of $\tilde{A}$. Furthermore, employing elementary transformations it is demonstrated that (4.18) is equivalent to (4.16).

From (4.15), (4.18) it follows that the solution of the problem (1.1), (1.2) has a nonsingular character.

In solitonic sector of the model ($b(\lambda) = 0$) one can be found the expression for the formulas of the reconstruction through eigenfunction of the discreet spectrum. can find the expression in terms of the eigenfunctions of discrete spectrum. For this purpose we first resolve the right of (4.8) at $\lambda \to 1$ and after equating it to $\tilde{\Phi}^0_1$ we will obtain from (4.15) the following:

$$u(x,y) = 2\ln\frac{1-\sqrt{1-(|\xi|^2+|\eta|^2)^2}}{|\xi|^2+|\eta|^2}, \tag{4.19}$$

where

$$\xi = \xi(x,y) = \sum_{n=1}^{2N} \frac{\bar{\beta}_n e^{-2ik(\bar{\lambda}_n)x}}{\bar{\lambda}_n + 1} \bar{\tilde{\phi}}^{-}{}_{11}(\lambda_n),$$

$$\eta = \eta(x,y) = 1 + \sum_{n=1}^{2N} \frac{\bar{\beta}_n e^{-2ik(\bar{\lambda}_n)x}}{\bar{\lambda}_n + 1} \bar{\tilde{\phi}}^{-}{}_{21}(\lambda_n),$$

$$\lambda_{n+N} = -\frac{1}{\bar{\lambda}_n}, \ \beta_{n+N} = -\frac{\bar{\beta}_n}{\bar{\lambda}_n^{\,2}}, \ \ n = 1, 2, ...N.$$

The functions $\tilde{\phi}_{11}^{-}(x, \lambda_n)$, $\tilde{\phi}_{21}^{-}(x, \lambda_n)$, involved in (4.19) can be found from the system of linear algebraic equations (4.9)-(4.11). Then:

$$\xi =< \bar{\Lambda}\bar{E}, \ R^{-1}E >_{2n}, \ \ \eta = 1- < \bar{\Lambda}\bar{E}, R^{-1}A\bar{E} >_{2n}, \tag{4.20}$$

where:
$A = \{A_{mn}\}$, $A_{mn} = E_m\bar{E}_n/(\lambda_m + \bar{\lambda}_n)$, $E_m = \beta_m^{1/2} e^{ik(\lambda_m)x}$,
$\Lambda = diag((\lambda_1 + 1)^{-1}, ...(\lambda_{2n} + 1)^{-1})$, $m, \ n = 1, ...2N$, $E = (E_1, ...E_{2N})$,
$R = I + A\bar{A}$.

N-"soliton" solution (4.19), (4.20) is the exact solution of the initial boundary problem. However in the simplest case ($N = 1$) it is it is rather cumbersome and it is beyond the paper.

Also it should be noted that the asymptotics of the solution for the continuous spectrum has been calculated in [6] proceeding from gauge equivalence between the given problem and the problem for Heisenberg's ferromagnet, with the technique of the ref.[7] being used essentially.

5. Application to the model of one-charged plasma at negative temperatures.

This model is lead by the model of the equilibrium system of charged particle (vortices) on the plane [2, 3]. The latter permits generalization of Debay-Hukkel classical model [8] for plasma or an electrolyte solution to the case of a nontrivial statistical "phone".

Following [2], [3], let us suppose that there is the Coulomb's system of N_0^+ particles possessing a charge $+e$ and N_0^- particles possessing a charge$-e$ embedded in some finite volume $V = L^2$ (where L is the linear scale). Furthermore, we will suppose that a temperatures of the ionic component coincides with a temperature of electronic one, with both of them being same and equal to T (Boltsmann's constant is considered to be equal to unit). Using the BBGKI chain it is convenient to consider a scalar potential $\Phi = \Phi(x, y)$:

$$e\Phi(x,y) = n_0 \int dx'dy'\phi(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}'), \ \ \mathbf{r} = (x,y), \tag{5.1}$$

where n_0 is the average density of charge particles, $\phi(\mathbf{r} - \mathbf{r}')$ is Coulomb's two-particles interaction, $\rho(\mathbf{r})$ is a difference between the ionic density and an electron one. Then employing rather rigorous procedure of the reduction of the chain we can obtain the equation:

$$\triangle\Phi = \frac{4\pi n_0 c}{l^0}[\exp(\frac{e\Phi}{T}) - \exp(-\frac{e\Phi}{T})], \tag{5.2}$$

where l^0 is a characteristic length,

$$c = \frac{V}{\int dxdy[\exp[\frac{n_0}{T}\int dx'dy'\phi(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')]]}.$$

Setting $u = \frac{e\Phi}{T}$ and Debay's radius $\tilde{\triangle} = (4\lambda_D^2/c)\triangle,\ \lambda_D^2 = l^0|T|/(8\pi n_0 e^2)$ in (5.2), we find:

$$sign(T)\tilde{\triangle}u = 4\sinh u. \tag{5.3}$$

This equation leads to the equation (1.1) at $T < 0$ (the case $T > 0$ is considered in [9, 10]).

According to the statement of the problem describe above the equation (5.3) held for $\mathbb{R}_+^2$ should be added by the given value of the field at the boundary $\partial\mathbb{R}_+^2$ considering.

Now we consider some physical aspects of the results obtained in the previous sections. Assuming $n_0 = N_0^{\pm}/V = const$, i.e in the thermodynamical limit.

For this purpose we will reveal some features of the behaviour of the equation (4.19). The total energy of the finite system of discrete isolated charges for the two-dimensional case [2], [3] is:

$$E_{tot} = \frac{1}{2}\sum_{i,j}(N_{0_i}^+ - N_{0_i}^-)\Phi_{ij}(N_{0_j}^+ - N_{0_j}^-), \tag{5.4}$$

where $N_0^+ = \sum_i N_{0_i}^+,\ N_0^- = \sum_j N_{0_j}^-,\ \Phi_{ij} = -(2e^2/l^0)\ln|\mathbf{r_i} - \mathbf{r_j}|$ is Coulomb's potential of the interaction between the particles of "i" and "j" clusters, $\mathbf{r_i}$ and $\mathbf{r_j}$ are their radius-vectors.

Comparing the expressions for Φ_{ij} with (4.19), we can see they have the similar form. Let $B = |\xi|^2 + |\eta|^2$, and $0 \le B \le 1$, due to the potential is real. Then from (4.19) it follows, that $u(x, y) \simeq 2\ln(B/2) \to -\infty$ at $B \to 0$ and $u(x, y) \to 0$ at $B \to 1$. Here in the terms of eigenfunctions of the discrete spectrum and values ξ, η, the first condition means that: $\xi,\ \eta \to 0$, which leads to $\sum_{n=1}^{2N}[\bar{\beta}_n \exp(-2ik(\bar{\lambda}_n)x)/(\bar{\lambda}_n + 1)]\tilde{\phi}_{21}^-(\lambda_n) \to -1$; it is possible on the case of finite $\mathbf{r}$. The latter can be fulfilled, in particular, at $\xi \to 0,\ \eta \to 1$, i.e. at $r \to \infty$.

Thus, we have been obtained the following qualitative picture. Let us suppose there is no external field first $(u(x, 0) = u_y(x, 0) = 0)$. Also clusters of the same charges can by formed from the charged particle. After appearance of an external field the space distribution being available before changes into another stationary state. At the any point $(x, y) \in \mathbb{R}_+^2$ the potential is the sum of potential arisen by the the boundary $\partial\mathbb{R}_+^2$ and potential of system; as result, some self-consistent field is created. The potential of the field is described by expression $Tu(x, y)/e$, where $u(x, y) = u(B)$, and the function $B = B(x, y)$ plays the role of the "transmission" function of the medium. At certain finite x, y a local and deep minimum of the potential corresponds to the minimal value of B, with $u < 0$ at the vicinity of the minimum. Considering $T < 0$ a value Φ proves to to be big and positive. This fact shows a local concentration of particles of the identical sign. Due to non-singular character of (4.19) Φ keep be bounded up and it leads to a limitation of the energetic spectrum of the system. So we deal with an abnormal system where

the inversion of the level distribution is available, it means that the positive value of the energy together with negative temperatures leads to the higher levels (in average) become more preferred for the system than the lower ones. Furthermore, the charges distribution of clusters with same sign appears more apparently.

The condition $r \to \infty$ and $u(B) \to 0$ corresponds to the maximum of B, i.e. as we recede from the boundary the field created by those charges fads due to the interaction with own field of the medium. The damping is slower than in the similar situation when temperatures are positive [9,10]. This is not difficult to see on comparing the solution of corresponding linear problems:

$$\triangle u = \pm 4u, \quad u \to 0 \quad at \quad r \to \infty, \qquad (5.5),$$

where the sign "+" corresponds to system with $T > 0$, whereas "-" corresponds to negative temperatures. In the former case Green's function is proportional to McDonald's function of the zero index and has the asymptotic: $\sim (e^{-x}/\sqrt{x})[1 + O(1/x)]$, $x \to \infty$, and in the latter case it is proportional to Neemann's function of zero index that behaves like: $(\sin(x - \pi/4)/\sqrt{x})[1 + O(1/x)]$, $x \to \infty$. Therefore the slower damping of the field at $T < 0$ seems to be explained by weaker defense of the boundary due to an alternation of domains with the same charged particles.

The picture describe above is true everywhere on the semi-plane except of the certain directions defined evidently. In these cases B takes an intermediate values: $B \in [\epsilon, 1 - \epsilon]$, $\epsilon > 0$, and the potential is an oscillating function in those directions, i.e. there is the distant order in the medium. It means that a coherent structure of the distribution of an electric field and densities of a plasma distribution arises.

The author is grateful to V.D.Lipovski for useful discussions.

This work has been due to a financial support of The Russian Foundation for Fundamental Research (Projects 98-01-01063, 00-01-00480).

References

[1]. A.I.Bobenko, Uspechi Math. Nauk, **46**, 4(1991)(in Russian).

[2]. G.Goyce and D.Montgomeri, J.Plasma Phys., **10**, 1, 10(1973).

[3]. D.Montgomeri and G.Goyce, Phys.Fluids, **17**, 6, 1139(1974).

[4]. Yu.B.Rumer and M.Ch.Rivkin. Thermodynamics, statistical physics and kinetic. Moscow, Nauka (1977)(in Russian).

[5]. E.Sh.Gutshabash, Vestnik LGU, ser.Fisika, Chimia , **4**, 84(1990)(in Russian).

[6]. G.G. Varzugin, E.Sh.Gutshabash and V.D.Lipovski, Teoreticheskaya i Matematicheskaya Fizika, **104**, 3, 513(1995)(in Russian).

[7]. P.Deift and X.Zhou, Ann.of Math., **137**, 2, 295(1993).

[8]. L.D.Landau and E.M.Lifchits. Statistical Physics. Moscow, Nauka,1982 (in Russian).

[9]. V.D.Lipovski and S.S.Nikulichev, Vestnik LGU, ser.Phisica, Chimia **4**, 61(1989)(in Russian).

[10]. E.Sh.Gutshabash, V.D.Lipovski and S.S. Nikulichev, Teoreticheskaya i Matematicheskaya Fizika, **115**, 323(1998)(in Russian); nlin.SI/0001012.